Class 11 Maths NCERT Solutions Chapter - 13

Limits and Derivatives Class 11

Chapter 13 Limits and Derivatives Exercise 13.1, 13.2, miscellaneous Solutions

Exercise 13.1 : Solutions of Questions on Page Number : **301**

Q1 :

Evaluate the Given limit: $\lim_{x\to 3} x+3$

Answer :

$$\lim_{x\to 3} x+3 = 3+3 = 6$$

Q2 :

Evaluate the Given limit: $\lim_{x\to \pi}\left(x-\frac{22}{7}\right)$

Answer :

$$\lim_{x\to \pi}\left(x-\frac{22}{7}\right) = \left(\pi-\frac{22}{7}\right)$$

Q3 :

Evaluate the Given limit: $\lim_{r\to 1}\pi r^2$

Answer :

$$\lim_{r\to 1}\pi r^2 = \pi(1)^2 = \pi$$

Q4 :

Evaluate the Given limit: $\lim_{x\to 4}\frac{4x+3}{x-2}$

Answer :

$$\lim_{x\to4}\frac{4x+3}{x-2}=\frac{4(4)+3}{4-2}=\frac{16+3}{2}=\frac{19}{2}$$

Q5 :

Evaluate the Given limit: $\lim_{x\to-1}\frac{x^{10}+x^5+1}{x-1}$

Answer :

$$\lim_{x\to-1}\frac{x^{10}+x^5+1}{x-1}=\frac{(-1)^{10}+(-1)^5+1}{-1-1}=\frac{1-1+1}{-2}=-\frac{1}{2}$$

Q6 :

Evaluate the Given limit: $\lim_{x\to0}\frac{(x+1)^5-1}{x}$

Answer :

$$\lim_{x\to0}\frac{(x+1)^5-1}{x}$$

Put $x + 1 = y$ so that $y \to 1$ as $x \to 0$.

$$\text{Accordingly, } \lim_{x\to0}\frac{(x+1)^5-1}{x}=\lim_{y\to1}\frac{y^5-1}{y-1}$$

$$=\lim_{y\to1}\frac{y^5-1^5}{y-1}$$

$$=5.1^{5-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=5$$

$$\therefore \lim_{x\to0}\frac{(x+5)^5-1}{x}=5$$

Q7 :

Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Answer :

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$
$$=\lim_{x\to 2}\frac{3x+5}{x+2}$$
$$=\frac{3(2)+5}{2+2}$$
$$=\frac{11}{4}$$

Q8 :

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Answer :

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$
$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$
$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$
$$=\frac{6\times 18}{7}$$
$$=\frac{108}{7}$$

Q9 :

Evaluate the Given limit: $\lim_{x \to 0} \frac{ax+b}{cx+1}$

Answer :

$$\lim_{x \to 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

Q10 :

Evaluate the Given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Answer :

$$\lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At $z = 1$, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}} = x$ so that $z \to 1$ as $x \to 1$.

$$\text{Accordingly, } \lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = \lim_{x \to 1} \frac{x^2-1}{x-1}$$

$$= \lim_{x \to 1} \frac{x^2-1^2}{x-1}$$

$$= 2.1^{2-1} \qquad \left[\lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 2$$

$$\therefore \lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = 2$$

Q11 :

Evaluate the Given limit: $\lim_{x \to 1} \frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c \neq 0$

Answer :

$$\lim_{x \to 1} \frac{ax^2+bx+c}{cx^2+bx+a} = \frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$
$$= \frac{a+b+c}{a+b+c}$$
$$= 1 \qquad [a+b+c \neq 0]$$

Q12 :

Evaluate the Given limit: $\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Answer :

$$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At $x = -2$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \lim_{x \to -2} \frac{\left(\frac{2+x}{2x}\right)}{x+2}$$
$$= \lim_{x \to -2} \frac{1}{2x}$$
$$= \frac{1}{2(-2)} = \frac{-1}{4}$$

Q13 :

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{bx}$

Answer :

$$\lim_{x\to 0}\frac{\sin ax}{bx}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\begin{aligned}
\text{Now, } \lim_{x\to 0}\frac{\sin ax}{bx} &= \lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx} \\
&= \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\left(\frac{a}{b}\right) \\
&= \frac{a}{b}\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right) && [x\to 0 \Rightarrow ax\to 0] \\
&= \frac{a}{b}\times 1 && \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right] \\
&= \frac{a}{b}
\end{aligned}$$

Q14 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$

Answer :

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

$$= \left(\frac{a}{b}\right)\times\frac{\lim\limits_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim\limits_{bx\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l} x\to 0 \Rightarrow ax\to 0 \\ \text{and } x\to 0\Rightarrow bx\to 0\end{array}\right]$$

$$= \left(\frac{a}{b}\right)\times\frac{1}{1} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y} = 1\right]$$

$$= \frac{a}{b}$$

Q15 :

Evaluate the Given limit: $\lim\limits_{x\to\pi}\frac{\sin(\pi - x)}{\pi(\pi - x)}$

Answer :

$$\lim_{x\to\pi}\frac{\sin(\pi - x)}{\pi(\pi - x)}$$

It is seen that $x \to \pi \Rightarrow (\pi - x) \to 0$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi - x)}{\pi(\pi - x)} = \frac{1}{\pi}\lim_{(\pi - x)\to 0}\frac{\sin(\pi - x)}{(\pi - x)}$$

$$= \frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y} = 1\right]$$

$$= \frac{1}{\pi}$$

Q16 :

Evaluate the given limit: $\lim\limits_{x\to 0}\frac{\cos x}{\pi - x}$

Answer :

$$\lim_{x\to 0}\frac{\cos x}{\pi - x}=\frac{\cos 0}{\pi - 0}=\frac{1}{\pi}$$

Q17 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Answer :

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1} \qquad \left[\cos x=1-2\sin^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Q18 :

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Answer :

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\begin{aligned}\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x} &= \frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}\\ &= \frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)\\ &= \frac{1}{b}\times\frac{1}{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)\\ &= \frac{1}{b}\times(a+\cos 0) \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\ &= \frac{a+1}{b}\end{aligned}$$

Q19 :

Evaluate the Given limit: $\lim\limits_{x\to 0} x\sec x$

Answer :

$$\lim_{x\to 0} x\sec x = \lim_{x\to 0}\frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

Q20 :

Evaluate the Given limit: $\lim\limits_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}\ \ a,b,a+b\neq 0$

Answer :

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax+bx}{ax+bx\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{\left(\lim_{ax\to 0}\frac{\sin ax}{ax}\right)\times\lim_{x\to 0}(ax)+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx\left(\lim_{bx\to 0}\frac{\sin bx}{bx}\right)} \qquad [\text{As } x\to 0\Rightarrow ax\to 0 \text{ and } bx\to 0]$$

$$=\frac{\lim_{x\to 0}(ax)+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$$

$$=\lim_{x\to 0}(1)$$

$$=1$$

Q21 :

Evaluate the Given limit: $\lim_{x\to 0}(\operatorname{cosec} x-\cot x)$

Answer :

At x = 0, the value of the given function takes the form $\infty-\infty$.

Now,

$$\lim_{x\to 0}(\operatorname{cosec} x-\cot x)$$

$$=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$=\frac{\lim\limits_{x\to 0}\frac{1-\cos x}{x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}$$

$$=\frac{0}{1}\qquad\left[\lim_{x\to 0}\frac{1-\cos x}{x}=0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=0$$

Q22 :

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Answer :

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x=\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x-\frac{\pi}{2}=y$ so that $x\to\frac{\pi}{2},\ y\to 0$.

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{y \to 0} \frac{\tan 2\left(y + \frac{\pi}{2}\right)}{y}$$

$$= \lim_{y \to 0} \frac{\tan(\pi + 2y)}{y}$$

$$= \lim_{y \to 0} \frac{\tan 2y}{y} \qquad [\tan(\pi + 2y) = \tan 2y]$$

$$= \lim_{y \to 0} \frac{\sin 2y}{y \cos 2y}$$

$$= \lim_{y \to 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right)$$

$$= \left(\lim_{2y \to 0} \frac{\sin 2y}{2y} \right) \times \lim_{y \to 0} \left(\frac{2}{\cos 2y} \right) \qquad [y \to 0 \Rightarrow 2y \to 0]$$

$$= 1 \times \frac{2}{\cos 0} \qquad \left[\lim_{x \to 0} \frac{\sin x}{x} = 1 \right]$$

$$= 1 \times \frac{2}{1}$$

$$= 2$$

Q23 :

Find $\lim\limits_{x \to 0} f(x)$ **and** $\lim\limits_{x \to 1} f(x)$**, where** $f(x) = \begin{cases} 2x + 3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

Answer :

The given function is

$$f(x) = \begin{cases} 2x + 3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} [2x + 3] = 2(0) + 3 = 3$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 3$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1} 3(x+1) = 3(1+1) = 6$$
$$\therefore \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = \lim_{x\to1} f(x) = 6$$

Q24 :

Find $\lim_{x\to1}$ ***f*(*x*), where *f*(*x*) =** $\begin{cases} x^2-1, & x\le 1 \\ -x^2-1, & x>1 \end{cases}$

Answer :

The given function is

$$f(x) = \begin{cases} x^2-1, & x\le 1 \\ -x^2-1, & x>1 \end{cases}$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1}\left[x^2-1\right] = 1^2-1 = 1-1 = 0$$
$$\lim_{x\to1^+} f(x) = \lim_{x\to1}\left[-x^2-1\right] = -1^2-1 = -1-1 = -2$$

It is observed that $\lim_{x\to1^-} f(x) \ne \lim_{x\to1^+} f(x)$.

Hence, $\lim_{x\to1} f(x)$ does not exist.

Q25 :

Evaluate $\lim_{x\to0}$ ***f*(*x*), where *f*(*x*) =** $\begin{cases} \dfrac{|x|}{x}, & x\ne 0 \\ 0, & x=0 \end{cases}$

Answer :

The given function is

f(*x*) = $\begin{cases} \dfrac{|x|}{x}, & x\ne 0 \\ 0, & x=0 \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left[\frac{|x|}{x}\right]$$

$$= \lim_{x \to 0} \left(\frac{-x}{x}\right) \qquad \left[\text{When } x \text{ is negaitve, } |x| = -x\right]$$

$$= \lim_{x \to 0} (-1)$$

$$= -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left[\frac{|x|}{x}\right]$$

$$= \lim_{x \to 0} \left[\frac{x}{x}\right] \qquad \left[\text{When } x \text{ is positive, } |x| = x\right]$$

$$= \lim_{x \to 0} (1)$$

$$= 1$$

It is observed that $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Q26 :

Find $\lim_{x \to 0}$ *f*(*x*), where *f*(*x*) = $\begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer :

The given function is

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left[\frac{x}{|x|}\right]\\ &= \lim_{x\to 0}\left[\frac{x}{-x}\right] \qquad \left[\text{When } x<0,\ |x|=-x\right]\\ &= \lim_{x\to 0}(-1)\\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]\\ &= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad \left[\text{When } x>0,\ |x|=x\right]\\ &= \lim_{x\to 0}(1)\\ &= 1\end{aligned}$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Q27 :

Find $\lim_{x\to 5}$ *f*(*x*), where *f*(*x*) = $|x|-5$

Answer :

The given function is $f(x)$ = $|x|-5$.

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}\left[|x|-5\right]$$
$$= \lim_{x\to 5}(x-5) \qquad [\text{When } x>0,\ |x|=x]$$
$$= 5-5$$
$$= 0$$
$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\left(|x|-5\right)$$
$$= \lim_{x\to 5}(x-5) \qquad [\text{When } x>0,\ |x|=x]$$
$$= 5-5$$
$$= 0$$
$$\therefore \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 0$$
$$\text{Hence, } \lim_{x\to 5} f(x) = 0$$

Q28 :

Suppose *f*(*x*) = $\begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$ **and if** $\lim\limits_{x\to 1}$ ***f*(*x*) = *f*(1) what are possible values of *a* and *b*?**

Answer :

The given function is

$$f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(a+bx) = a+b$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(b-ax) = b-a$$
$$f(1) = 4$$
$$\text{It is given that } \lim_{x\to 1} f(x) = f(1).$$
$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$$
$$\Rightarrow a+b=4 \text{ and } b-a=4$$
$$\text{On solving these two equations, we obtain } a=0 \text{ and } b=4.$$

Thus, the respective possible values of *a* and *b* are 0 and 4.

Q29 :

Let $a_1,\ a_2,\ \ldots,\ a_n$ be fixed real numbers and define a function

$$f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$$

What is $\lim\limits_{x\to a_1}$ *f*(*x*)? For some $a\neq a_1,\ a_2\ldots,\ a_n,$ compute $\lim\limits_{x\to a}$ *f*(*x*).

Answer :

The given function is $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$

$$\begin{aligned}\lim_{x\to a_1} f(x)&=\lim_{x\to a_1}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]\\&=\left[\lim_{x\to a_1}(x-a_1)\right]\left[\lim_{x\to a_1}(x-a_2)\right]\ldots\left[\lim_{x\to a_1}(x-a_n)\right]\\&=(a_1-a_1)(a_1-a_2)\ldots(a_1-a_n)=0\end{aligned}$$

$$\therefore \lim_{x\to a_1} f(x)=0$$

$$\begin{aligned}\text{Now, }\lim_{x\to a} f(x)&=\lim_{x\to a}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]\\&=\left[\lim_{x\to a}(x-a_1)\right]\left[\lim_{x\to a}(x-a_2)\right]\ldots\left[\lim_{x\to a}(x-a_n)\right]\\&=(a-a_1)(a-a_2)\ldots\ldots(a-a_n)\end{aligned}$$

$$\therefore \lim_{x\to a} f(x)=(a-a_1)(a-a_2)\ldots(a-a_n)$$

Q30 :

If *f*(*x*) = $\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$**.**

For what value (s) of a does $\lim\limits_{x\to a}$ *f*(*x*) exists?

Answer :

The given function is

$$f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$$

When $a=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(|x|+1)$$
$$=\lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x|=-x]$$
$$=-0+1$$
$$=1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(|x|-1)$$
$$=\lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x|=x]$$
$$=0-1$$
$$=-1$$

Here, it is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When $a<0$,

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [x<a<0 \Rightarrow |x|=-x]$$
$$=-a+1$$

$$\lim_{x\to a^+} f(x)=\lim_{x\to a^+}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [a<x<0 \Rightarrow |x|=-x]$$
$$=-a+1$$

$$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=-a+1$$

Thus, limit of $f(x)$ exists at $x=a$, where $a<0$.

When $a>0$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x|-1)$$
$$= \lim_{x\to a}(x-1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a-1$$
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x|-1)$$
$$= \lim_{x\to a}(x-1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a-1$$
$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim\limits_{x\to a} f(x)$ exists for all $a \neq 0$.

Q31 :

If the function *f*(*x*) satisfies $\lim\limits_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim\limits_{x\to 1} f(x)$.

Answer :

$$\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$$
$$\Rightarrow \frac{\lim\limits_{x\to 1}(f(x)-2)}{\lim\limits_{x\to 1}(x^2-1)} = \pi$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi \lim_{x\to 1}(x^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi(1^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - \lim_{x\to 1} 2 = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - 2 = 0$$
$$\therefore \lim_{x\to 1} f(x) = 2$$

Q32 :

If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$**. For what integers *m* and *n* does** $\lim_{x\to 0} f(x)$ **and** $\lim_{x\to 1} f(x)$ **exist?**

Answer :

The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0}\left(mx^2+n\right) \\ &= m(0)^2+n \\ &= n\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0}\left(nx+m\right) \\ &= n(0)+m \\ &= m.\end{aligned}$$

Thus, $\lim_{x\to 0} f(x)$ exists if *m* = *n*.

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1}\left(nx+m\right) \\ &= n(1)+m \\ &= m+n\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1}\left(nx^3+m\right) \\ &= n(1)^3+m \\ &= m+n\end{aligned}$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x).$$

Thus, $\lim_{x\to 1} f(x)$ exists for any integral value of *m* and *n*.

Exercise 13.2 : Solutions of Questions on Page Number : **312**

Q1 :

Find the derivative of x^2 - 2 at x = 10.

Answer :

Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$
$$= \lim_{h\to 0}\frac{\left[(10+h)^2-2\right]-(10^2-2)}{h}$$
$$= \lim_{h\to 0}\frac{10^2+2.10.h+h^2-2-10^2+2}{h}$$
$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$
$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Q2 :

Find the derivative of 99*x* at *x* = 100.

Answer :

Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$
$$= \lim_{h\to 0}\frac{99(100+h)-99(100)}{h}$$
$$= \lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}$$
$$= \lim_{h\to 0}\frac{99h}{h}$$
$$= \lim_{h\to 0}(99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Q3 :

Find the derivative of *x* at *x* = 1.

Answer :

Let $f(x) = x$.
Accordingly,

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0}\frac{(1+h)-1}{h} \\
&= \lim_{h\to 0}\frac{h}{h} \\
&= \lim_{h\to 0}(1) \\
&= 1
\end{aligned}
$$

Thus, the derivative of x at $x = 1$ is 1.

Q4 :

Find the derivative of the following functions from first principle.

(i) x^3 - 27 (ii) (x - 1) (x - 2)

(ii) $\frac{1}{x^2}$ **(iv)** $\frac{x+1}{x-1}$

Answer :

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h} \\
&= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h} \\
&= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h} \\
&= \lim_{h\to 0}\left(h^2+3x^2+3xh\right) \\
&= 0+3x^2+0 = 3x^2
\end{aligned}
$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\
&= \lim_{h \to 0} \frac{\left(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2\right) - \left(x^2 - 2x - x + 2\right)}{h} \\
&= \lim_{h \to 0} \frac{\left(hx + hx + h^2 - 2h - h\right)}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2 - 3h}{h} \\
&= \lim_{h \to 0} (2x + h - 3) \\
&= (2x + 0 - 3) \\
&= 2x - 3
\end{aligned}
$$

(iii) Let $f(x) = \frac{1}{x^2}$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2 (x+h)^2} \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2 (x+h)^2} \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2 (x+h)^2} \right] \\
&= \lim_{h \to 0} \left[\frac{-h - 2x}{x^2 (x+h)^2} \right] \\
&= \frac{0 - 2x}{x^2 (x+0)^2} = \frac{-2}{x^3}
\end{aligned}
$$

(iv) Let $f(x) = \frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$
$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Q5 :

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Answer :

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\ldots+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\ldots+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+\ldots+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+\ldots+x+1$$

At $x=0$,

$$f'(0)=1$$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+\ldots+1+1=[1+1+\ldots+1+1]_{100\text{ terms}}=1\times100=100$$

Thus, $f'(1)=100\times f^1(0)$

Q6 :

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$ for some fixed real number *a*.

Answer :

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$

$$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n\right)$$

$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+\ldots+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\ldots+a^{n-1}+a^n(0)$$

$$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\ldots+a^{n-1}$$

Q7 :

For some constants *a* and *b*, find the derivative of

(i) (*x* - *a*) (*x* - *b*) (ii) (ax^2 + *b*)^2 (iii) $\frac{x-a}{x-b}$

Answer :

(i) Let *f* (*x*) = (*x* - *a*) (*x* - *b*)

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}\left(x^2 - (a+b)x + ab\right)$$

$$= \frac{d}{dx}\left(x^2\right) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let $f(x) = \left(ax^2 + b\right)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}\left(a^2x^4 + 2abx^2 + b^2\right) = a^2\frac{d}{dx}\left(x^4\right) + 2ab\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(b^2\right)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$f'(x) = a^2\left(4x^3\right) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax\left(ax^2 + b\right)$$

(iii) Let $f(x) = \frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$
$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$
$$=\frac{x-b-x+a}{(x-b)^2}$$
$$=\frac{a-b}{(x-b)^2}$$

Q8 :

Find the derivative of $\frac{x^n-a^n}{x-a}$ **for some constant *a*.**

Answer :

$$\text{Let } f(x)=\frac{x^n-a^n}{x-a}$$
$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$
$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$
$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Q9 :

Find the derivative of

(i) $2x-\frac{3}{4}$ **(ii) ($5x^3$ + 3*x* - 1) (*x* - 1)**

(iii) x^{-3} (5 + 3x) (iv) x^5 (3 - 6x^{-9})

(v) x^{-4} (3 - 4x^{-5}) (vi) $\frac{2}{x+1}-\frac{x^2}{3x-1}$

Answer :

(i) Let $f\left(x\right)=2x-\frac{3}{4}$

$$\begin{aligned}f'\left(x\right)&=\frac{d}{dx}\left(2x-\frac{3}{4}\right)\\&=2\frac{d}{dx}\left(x\right)-\frac{d}{dx}\left(\frac{3}{4}\right)\\&=2-0\\&=2\end{aligned}$$

(ii) Let f (x) = (5x^3 + 3x - 1) (x - 1)

By Leibnitz product rule,

$$\begin{aligned}f'\left(x\right)&=\left(5x^3+3x-1\right)\frac{d}{dx}\left(x-1\right)+\left(x-1\right)\frac{d}{dx}\left(5x^3+3x-1\right)\\&=\left(5x^3+3x-1\right)\left(1\right)+\left(x-1\right)\left(5.3x^2+3-0\right)\\&=\left(5x^3+3x-1\right)+\left(x-1\right)\left(15x^2+3\right)\\&=5x^3+3x-1+15x^3+3x-15x^2-3\\&=20x^3-15x^2+6x-4\end{aligned}$$

(iii) Let$f$ (x) = x^{-3} (5 + 3x)

By Leibnitz product rule,

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x) + (5+3x)\frac{d}{dx}(x^{-3})$$
$$= x^{-3}(0+3) + (5+3x)(-3x^{-3-1})$$
$$= x^{-3}(3) + (5+3x)(-3x^{-4})$$
$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$
$$= -6x^{-3} - 15x^{-4}$$
$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$
$$= \frac{-3x^{-3}}{x}(2x+5)$$
$$= \frac{-3}{x^4}(5+2x)$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5\frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9})\frac{d}{dx}(x^5)$$
$$= x^5\left\{0-6(-9)x^{-9-1}\right\} + (3-6x^{-9})(5x^4)$$
$$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$$
$$= 54x^{-5} + 15x^4 - 30x^{-5}$$
$$= 24x^{-5} + 15x^4$$
$$= 15x^4 + \frac{24}{x^5}$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5})\frac{d}{dx}(x^{-4})$$
$$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\} + (3-4x^{-5})(-4)x^{-4-1}$$
$$= x^{-4}(20x^{-6}) + (3-4x^{-5})(-4x^{-5})$$
$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$
$$= 36x^{-10} - 12x^{-5}$$
$$= -\frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x)=\frac{d}{dx}\left(\frac{2}{x+1}\right)-\frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x)=\left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right]-\left[\frac{(3x-1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$=\left[\frac{(x+1)(0)-2(1)}{(x+1)^2}\right]-\left[\frac{(3x-1)(2x)-(x^2)(3)}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$$

Q10 :

Find the derivative of cos *x* from first principle.

Answer :

Let *f* (*x*) = cos *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h) - \sin x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= -\cos x \left(\lim_{h \to 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \to 0} \left(\frac{\sin h}{h} \right)$$

$$= -\cos x (0) - \sin x (1) \qquad \left[\lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \to 0} \frac{\sin h}{h} = 1 \right]$$

$$= -\sin x$$

$$\therefore f'(x) = -\sin x$$

Q11 :

Find the derivative of the following functions:

(i) sin *x* cos *x* (ii) sec *x* (iii) 5 sec *x* + 4 cos *x*

(iv) cosec *x* (v) 3cot *x* + 5cosec *x*

(vi) 5sin *x* - 6cos *x* + 7 (vii) 2tan *x* - 7sec *x*

Answer :

(i) Let *f* (*x*) = sin *x* cos *x*. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h} \\
&= \lim_{h \to 0} \frac{1}{2h}\left[2\sin(x+h)\cos(x+h) - 2\sin x \cos x\right] \\
&= \lim_{h \to 0} \frac{1}{2h}\left[\sin 2(x+h) - \sin 2x\right] \\
&= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}\right] \\
&= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2} \sin\frac{2h}{2}\right] \\
&= \lim_{h \to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right] \\
&= \lim_{h \to 0} \cos(2x+h) . \lim_{h \to 0} \frac{\sin h}{h} \\
&= \cos(2x+0).1 \\
&= \cos 2x
\end{aligned}
$$

(ii) Let*f* (x) = sec x. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)} \\
&= \frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
&= \frac{1}{\cos x}.1.\frac{\sin x}{\cos x} \\
&= \sec x\tan x
\end{aligned}
$$

(iii) Let $f(x) = 5 \sec x + 4 \cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5\sec(x+h) + 4\cos(x+h) - [5\sec x + 4\cos x]}{h}$$

$$= 5\lim_{h \to 0} \frac{[\sec(x+h) - \sec x]}{h} + 4\lim_{h \to 0} \frac{[\cos(x+h) - \cos x]}{h}$$

$$= 5\lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right] + 4\lim_{h \to 0} \frac{1}{h}[\cos(x+h) - \cos x]$$

$$= 5\lim_{h \to 0} \frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)}\right] + 4\lim_{h \to 0} \frac{1}{h}[\cos x \cos h - \sin x \sin h - \cos x]$$

$$= \frac{5}{\cos x} \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right] + 4\lim_{h \to 0} \frac{1}{h}[-\cos x(1-\cos h) - \sin x \sin h]$$

$$= \frac{5}{\cos x} . \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right] + 4\left[-\cos x \lim_{h \to 0} \frac{(1-\cos h)}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}\right]$$

$$= \frac{5}{\cos x} . \lim_{h \to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right) . \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right] + 4[(-\cos x).(0) - (\sin x).1]$$

$$= \frac{5}{\cos x} . \left[\lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} . \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] - 4\sin x$$

$$= \frac{5}{\cos x} . \frac{\sin x}{\cos x} . 1 - 4\sin x$$

$$= 5\sec x \tan x . - 4\sin x$$

(iv) Let $f(x) = \text{cosec } x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{3\cot(x+h)+5\operatorname{cosec}(x+h)-3\cot x-5\operatorname{cosec} x}{h}\\
&= 3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right] \qquad \ldots(1)
\end{aligned}
$$

$$
\begin{aligned}
\text{Now, } &\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]\\
&= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)\\
&= -1.\frac{1}{\sin x\cdot\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \qquad \ldots(2)
\end{aligned}
$$

$$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x \qquad \ldots(3)$$

$$f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec}x\cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[5\{\sin(x+h) - \sin x\} - 6\{\cos(x+h) - \cos x\}\right]$$
$$= 5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h) - \sin x\right] - 6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h) - \cos x\right]$$
$$= 5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right] - 6\lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= 5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right] - 6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h) - \sin x\sin h}{h}\right]$$
$$= 5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) - 6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h} - \frac{\sin x\sin h}{h}\right]$$
$$= 5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right] - 6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right) - \sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$
$$= 5\cos x.1 - 6\left[(-\cos x).(0) - \sin x.1\right]$$
$$= 5\cos x + 6\sin x$$

(vii) Let $f(x) = 2\tan x - 7\sec x$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h) - 7\sec(x+h) - 2\tan x + 7\sec x\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\} - 7\left\{\sec(x+h)-\sec x\right\}\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos x\cos(x+h)}\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos x\cos(x+h)}\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right] \\
&= 2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right] \\
&= 2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right) - 7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right) \\
&= 2.1.\frac{1}{\cos x\cos x} - 7.1\left(\frac{\sin x}{\cos x\cos x}\right) \\
&= 2\sec^2 x - 7\sec x\tan x
\end{aligned}
$$

Exercise Miscellaneous : Solutions of Questions on Page Number : **317**

Q1 :

Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x+1)$

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Answer :

(i) Let *f*(*x*) = -*x*. Accordingly, $f(x+h) = -(x+h)$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0}\frac{-x-h+x}{h} \\ &= \lim_{h\to 0}\frac{-h}{h} \\ &= \lim_{h\to 0}(-1) = -1 \end{aligned}$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right] \end{aligned}$$

$$\begin{aligned} &= \lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right] \\ &= \lim_{h\to 0}\frac{1}{x(x+h)} \\ &= \frac{1}{x\cdot x} = \frac{1}{x^2} \end{aligned}$$

(iii) Let *f*(*x*) = sin (*x* + 1). Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]\\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]\\
&= \lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} && \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]\\
&= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1 && \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\
&= \cos(x+1)
\end{aligned}
$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]
\end{aligned}
$$

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

$$= -\sin\left(x-\frac{\pi}{8}\right)$$

Q2 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*x* + *a*)

Answer :

Let $f(x) = x + a$. Accordingly, $f(x+h) = x+h+a$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{x+h+a-x-a}{h}\\
&= \lim_{h\to 0}\left(\frac{h}{h}\right)\\
&= \lim_{h\to 0}(1)\\
&= 1
\end{aligned}
$$

Q3 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(px+q)\left(\frac{r}{x}+s\right)$

Answer :

$$\text{Let } f(x) = (px+q)\left(\frac{r}{x}+s\right)$$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)'\\
&= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p)\\
&= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p\\
&= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p\\
&= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps\\
&= ps - \frac{qr}{x^2}
\end{aligned}
$$

Q4 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*ax* + *b*) (*cx* + *d*)^2

Answer :

Let $f(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= (ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b)\\
&= (ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right)+(cx+d)^2\frac{d}{dx}(ax+b)\\
&= (ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]\\
&= (ax+b)\left(2c^2x+2cd\right)+\left(cx+d^2\right)a\\
&= 2c(ax+b)(cx+d)+a(cx+d)^2
\end{aligned}
$$

Q5 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{ax+b}{cx+d}$

Answer :

Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$
\begin{aligned}
f'(x) &= \frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}\\
&= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}\\
&= \frac{acx+ad-acx-bc}{(cx+d)^2}\\
&= \frac{ad-bc}{(cx+d)^2}
\end{aligned}
$$

Q6 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Answer :

$$\text{Let } f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}, \text{ where } x\neq 0$$

By quotient rule,

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{x-1-x-1}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{-2}{(x-1)^2},\ x\neq 0,\ 1$$

Q7 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{1}{ax^2+bx+c}$

Answer :

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{\left(ax^2+bx+c\right)\frac{d}{dx}(1)-\frac{d}{dx}\left(ax^2+bx+c\right)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{\left(ax^2+bx+c\right)(0)-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

Q8 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{ax+b}{px^2+qx+r}$

Answer :

$$\text{Let } f(x)=\frac{ax+b}{px^2+qx+r}$$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

Q9 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{px^2+qx+r}{ax+b}$

Answer :

$$\text{Let } f(x)=\frac{px^2+qx+r}{ax+b}$$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)-\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-\left(px^2+qx+r\right)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Q10 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Answer :

$$\text{Let } f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x) \qquad \left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Q11 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $4\sqrt{x}-2$

Answer :

$$\text{Let } f(x) = 4\sqrt{x}-2$$

$$f'(x) = \frac{d}{dx}\left(4\sqrt{x}-2\right) = \frac{d}{dx}\left(4\sqrt{x}\right) - \frac{d}{dx}(2)$$

$$= 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0 = 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= \left(2x^{-\frac{1}{2}}\right) = \frac{2}{\sqrt{x}}$$

Q12 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax+b)^n$

Answer :

Let $f(x) = (ax+b)^n$. Accordingly, $f(x+h) = \{a(x+h)+b\}^n = (ax+ah+b)^n$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h} \\
&= \lim_{h\to 0}\frac{(ax+b)^n\left(1+\dfrac{ah}{ax+b}\right)^n-(ax+b)^n}{h} \\
&= (ax+b)^n\lim_{h\to 0}\frac{\left(1+\dfrac{ah}{ax+b}\right)^n-1}{h} \\
&= (ax+b)^n\lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\underline{|2}}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad(\text{Using binomial theorem}) \\
&= (ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{\underline{|2}(ax+b)^2}+\ldots(\text{Terms containing higher degrees of } h)\right] \\
&= (ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{\underline{|2}(ax+b)^2}+\ldots\right] \\
&= (ax+b)^n\left[\frac{na}{(ax+b)}+0\right] \\
&= na\frac{(ax+b)^n}{(ax+b)} \\
&= na(ax+b)^{n-1}
\end{aligned}
$$

Q13 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax + b)^n$ $(cx + d)^m$

Answer :

Let $f(x)=(ax+b)^n(cx+d)^m$

By Leibnitz product rule,

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^m\frac{d}{dx}(ax+b)^n \qquad ...(1)$$

Now, let $f_1(x)=(cx+d)^m$

$$f_1(x+h)=(cx+ch+d)^m$$

$$\begin{aligned}
f_1'(x)&=\lim_{h\to0}\frac{f_1(x+h)-f_1(x)}{h}\\
&=\lim_{h\to0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}\\
&=(cx+d)^m\lim_{h\to0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]\\
&=(cx+d)^m\lim_{h\to0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+...\right)-1\right]\\
&=(cx+d)^m\lim_{h\to0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+...(\text{Terms containing higher degrees of } h)\right]\\
&=(cx+d)^m\lim_{h\to0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+...\right]\\
&=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]\\
&=\frac{mc(cx+d)^m}{(cx+d)}\\
&=mc(cx+d)^{m-1}
\end{aligned}$$

$$\frac{d}{dx}(cx+d)^m=mc(cx+d)^{m-1} \qquad ...(2)$$

Similarly, $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1} \qquad ...(3)$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned}
f'(x)&=(ax+b)^n\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^m\left\{na(ax+b)^{n-1}\right\}\\
&=(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]
\end{aligned}$$

Q14 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): sin (*x* + *a*)

Answer :

Let $f(x)=\sin(x+a)$

$f(x+h)=\sin(x+h+a)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]\\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]\\
&= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} && \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]\\
&= \cos\left(\frac{2x+2a}{2}\right)\times 1 && \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\
&= \cos(x+a)
\end{aligned}
$$

Q15 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): cosec *x* cot *x*

Answer :

Let $f(x)=\operatorname{cosec} x\cot x$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x(\cot x)' + \cot x(\operatorname{cosec} x)' \qquad ...(1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x+h) = \cot(x+h)$

By first principle,

$$\begin{aligned}
f_1'(x) &= \lim_{h\to 0} \frac{f_1(x+h) - f_1(x)}{h} \\
&= \lim_{h\to 0} \frac{\cot(x+h) - \cot x}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin x\cos(x+h) - \cos x\sin(x+h)}{\sin x\sin(x+h)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right] \\
&= \frac{1}{\sin x}.\lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right] \\
&= \frac{-1}{\sin x}.\left(\lim_{h\to 0} \frac{\sin h}{h}\right)\left(\lim_{h\to 0} \frac{1}{\sin(x+h)}\right) \\
&= \frac{-1}{\sin x}.1.\left(\frac{1}{\sin(x+0)}\right) \\
&= \frac{-1}{\sin^2 x} \\
&= -\operatorname{cosec}^2 x
\end{aligned}$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \qquad ...(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$\begin{aligned}
f_2'(x) &= \lim_{h\to 0} \frac{f_2(x+h) - f_2(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\operatorname{cosec}(x+h) - \operatorname{cosec} x\right]
\end{aligned}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \frac{1}{\sin x} . \lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} . \lim_{h\to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} . \lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} . \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} . \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} . \lim_{h\to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} . 1 . \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} . \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x . \cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x . \cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$

$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Q16 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\cos x}{1+\sin x}$

Answer :

Let $f(x)=\frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)}$$

Q17 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin x+\cos x}{\sin x-\cos x}$

Answer :

Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x) = \frac{(\sin x - \cos x)\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= \frac{-\left[\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x\right]}{(\sin x - \cos x)^2}$$

$$= \frac{-[1+1]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

Q18 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sec x - 1}{\sec x + 1}$

Answer :

Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}=\frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$=\frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$=\frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

Q19 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\sin^n x$

Answer :

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

$$\therefore \frac{dy}{dx}=\cos x,\ \text{i.e.,}\ \frac{d}{dx}\sin x=\cos x$$

For $n = 2$, $y = \sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x (\sin x)' \qquad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2\sin x \cos x \qquad ...(1)$$

For $n = 3$, $y = \sin^3 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$$

$$= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' \qquad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin^2 x + \sin x (2\sin x \cos x) \qquad \text{[Using (1)]}$$

$$= \cos x \sin^2 x + 2\sin^2 x \cos x$$

$$= 3\sin^2 x \cos x$$

We assert that $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

i.e., $\frac{d}{dx}(\sin^k x) = k\sin^{(k-1)} x \cos x \qquad ...(2)$

Consider

$$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^k x)$$

$$= (\sin x)' \sin^k x + \sin x (\sin^k x)' \qquad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin^k x + \sin x \left(k \sin^{(k-1)} x \cos x\right) \qquad \text{[Using (2)]}$$

$$= \cos x \sin^k x + k\sin^k x \cos x$$

$$= (k+1)\sin^k x \cos x$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Q20 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a + b\sin x}{c + d\cos x}$

Answer :

Let $f(x)=\dfrac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

Q21 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{\sin(x+a)}{\cos x}$

Answer :

Let $f(x)=\dfrac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)(-\sin x)}{\cos^2 x} \qquad \ldots \text{(i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\qquad\left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=\left(\cos\frac{2x+2a}{2}\right)\times 1\qquad\left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=\cos(x+a)\qquad\ldots\text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$=\frac{\cos(x+a-x)}{\cos^2 x}$$

$$=\frac{\cos a}{\cos^2 x}$$

Q22 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): x^4 (5 sin *x* - 3 cos *x*)

Answer :

Let $f(x)=x^4\left(5\sin x-3\cos x\right)$

By product rule,

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$
$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

Q23 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (x^2 + 1) cos *x*

Answer :

Let $f(x) = (x^2 + 1)\cos x$

By product rule,

$$f'(x) = (x^2 + 1)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1)$$
$$= (x^2 + 1)(-\sin x) + \cos x(2x)$$
$$= -x^2 \sin x - \sin x + 2x\cos x$$

Q24 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (ax^2 + sin *x*) (*p* + *q* cos *x*)

Answer :

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By product rule,

$$f'(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$
$$= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x)$$
$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

Q25 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(x+\cos x)(x-\tan x)$

Answer :

Let $f(x)=(x+\cos x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$
$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$
$$=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x) \qquad \text{... (i)}$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By first principle,

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right) \\
&= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \qquad \ldots \text{(ii)}
\end{aligned}$$

Therefore, from (i) and (ii), we obtain

$$\begin{aligned}
f'(x) &= (x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x) \\
&= (x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x) \\
&= -\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)
\end{aligned}$$

Q26 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{4x+5\sin x}{3x+7\cos x}$

Answer :

Let $f(x)=\dfrac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35\left(\cos^2 x+\sin^2 x\right)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Q27 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer :

Let $f(x)=\dfrac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\frac{\pi}{4}.\left[\frac{\sin x\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$=\cos\frac{\pi}{4}.\left[\frac{\sin x\cdot 2x-x^2\cos x}{\sin^2 x}\right]$$

$$=\frac{x\cos\frac{\pi}{4}\left[2\sin x-x\cos x\right]}{\sin^2 x}$$

Q28 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{1+\tan x}$

Answer :

Let $f(x)=\frac{x}{1+\tan x}$

$$f'(x)=\frac{(1+\tan x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x)=\frac{(1+\tan x)-x\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \text{... (i)}$$

Let $g(x)=1+\tan x$. Accordingly, $g(x+h)=1+\tan(x+h)$.

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1+\tan x)=\sec^2 x \qquad \text{... (ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

Q29 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*x* + sec *x*) (*x* - tan *x*)

Answer :

Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$

$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}\tan x=\sec^2 x \qquad \text{... (ii)}$$

$$
\begin{aligned}
f_2'(x) &= \lim_{h\to 0}\left(\frac{f_2(x+h)-f_2(x)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right]\\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]\\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]\\
&= \frac{1}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]\\
&= \sec x.\frac{\left\{\lim\limits_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim\limits_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim\limits_{h\to 0}\cos(x+h)}\\
&= \sec x.\frac{\sin x.1}{\cos x}
\end{aligned}
$$

$$
\Rightarrow \frac{d}{dx}\sec x = \sec x\tan x \qquad \text{.. (iii)}
$$

From (i), (ii), and (iii), we obtain

$$
f'(x) = (x+\sec x)\left(1-\sec^2 x\right)+(x-\tan x)(1+\sec x\tan x)
$$

Q30 :

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{\sin^n x}$

Answer :

Let $f(x)=\frac{x}{\sin^n x}$

By quotient rule,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx}\sin^n x=n\sin^{n-1} x\cos x$

Therefore,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

$$=\frac{\sin^n x.1-x\left(n\sin^{n-1} x\cos x\right)}{\sin^{2n} x}$$

$$=\frac{\sin^{n-1} x\left(\sin x-nx\cos x\right)}{\sin^{2n} x}$$

$$=\frac{\sin x-nx\cos x}{\sin^{n+1} x}$$